The changes in amplitude of short gravity waves on steady non-uniform currents

By M. S. LONGUET-HIGGINS
National Institute of Oceanography, Wormley, Surrey
AND R. W. STEWART
University of British Columbia, Vancouver

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The common assumption that the energy of waves on a non-uniform current **U** is propagated with a velocity $(\mathbf{U}+\mathbf{c}_g)$ where $\mathbf{c}_g$ is the group-velocity, and that no further interaction takes place, is shown in this paper to be incorrect. In fact the current does additional work on the waves at a rate $\gamma_{ij}S_{ij}$ where γ_{ij} is the symmetric rate-of-strain tensor associated with the current, and S_{ij} is the radiation stress tensor introduced earlier (Longuet-Higgins & Stewart 1960).

In the present paper we first obtain an asymptotic solution for the combined velocity potential in the simple case (1) when the non-uniform current **U** is in the direction of wave propagation and the horizontal variation of **U** is compensated by a vertical upwelling from below. The change in wave amplitude is shown to be such as would be found by inclusion of the radiation stress term.

In a second example (2) the current on the x-axis is assumed to be as in (1), but the horizontal variation in **U** is compensated by a small horizontal inflow from the sides. It is found that in that case the wave amplitude is also affected by the horizontal advection of wave energy from the sides.

From cases (1) and (2) the general law of interaction between short waves and non-uniform currents is inferred. This is then applied to a third example (3) when waves encounter a current with vertical axis of shear, at an oblique angle. The change in wave amplitude is shown to differ somewhat from the previously accepted value.

The conclusion that non-linear interactions affect the amplification of the waves has some bearing on the theoretical efficiency of hydraulic and pneumatic breakwaters.

1. Introduction

When short surface waves of any kind are propagated over the surface of a medium in steady but non-uniform motion, they tend to undergo refractive changes in length, direction and amplitude. The changes in length and direction depend on kinematical considerations only; a quite general treatment applicable to water waves has been given, for example, by Ursell (1960). But changes in the wave amplitude are less straightforward. Commonly (see Unna 1942; Suthons 1945; Johnson 1947; Evans 1955; Groen & Dorrestein 1958) it has been assumed without justification that no coupling between the waves and current takes place, and that the wave energy is simply propagated with a velocity equal to $(\mathbf{U}+\mathbf{c}_g)$, where $\mathbf{c}_g$ is the vector group-velocity and $\mathbf{U}$ the local stream velocity. On the contrary, in a recent paper (Longuet-Higgins & Stewart 1960; this paper will be referred to as I), it was found that short gravity waves, riding on the backs of longer waves, are modified to a much greater extent than would be predicted if there were no interchange of energy between the short and the long waves. The discrepancy may be attributed to a term in the equation of energy transfer, called by us the *radiation stress*, and previously overlooked. The stress term occurs quite generally, and must give rise to changes in the wave amplitude in other situations besides the particular one that was considered.

The purpose of the present paper is to study the changes in amplitude of gravity waves riding on steady but non-uniform currents. The subject is of special interest owing to its possible application to bubble-breakwaters, whose action is probably to be ascribed largely to the stopping power of a horizontal current opposing the waves (Taylor 1955; Evans 1955; Straub, Bowers & Tarrapore 1959). Ocean waves entering tidal streams or crossing river flows are known to be subject to a similar effect (Unna 1942; Johnson 1947). The following discussion will be limited to the case of deep currents, that is to say, those for which the change in current velocity in a vertical distance equal to the wavelength is small compared with the wave velocity itself. But quite similar results would apply to waves on shearing currents which penetrated to a depth of only a fraction of a wavelength.

In our first example we consider a system of waves superposed on a current which varies gradually in the x-direction (the direction of wave propagation), and in which the variation in surface current is made up by a vertical upwelling (or downwelling). The modification which the currents produce in the wave form is calculated rigorously by a perturbation method. It is found that, while the variation in the wave-number k is given by the expected formula

$$\frac{1}{k}\frac{\partial k}{\partial x} = -\frac{1}{c+2U}\frac{\partial U}{\partial x}, \tag{1.1}$$

the variation in the wave amplitude, on the other hand, is given by

$$\frac{1}{a}\frac{\partial a}{\partial x} = -\frac{2c+3U}{(c+2U)^2}\frac{\partial U}{\partial x}, \tag{1.2}$$

which is a higher rate of change than if there were no interaction between waves and currents. It is shown that this last result is consistent with the assumption that the equation governing the growth of wave energy E is

$$\frac{\partial}{\partial x}[E(c_g+U)] + S_x\frac{\partial U}{\partial x} = 0, \tag{1.3}$$

where S_x is the radiation stress mentioned earlier. (In deep water, $S_x = \frac{1}{2}E$.) This is to say that in addition to the transport of energy by the group-velocity and stream velocity, the current does work on the waves at a rate $S_x\partial U/\partial x$ per unit distance. In §4, this conclusion is shown also to be consistent with our earlier results in I concerning the steepening of surface waves on long waves or tidal streams. Integration of (1.3) leads to the result

$$a \propto [c(c+2U)]^{-\frac{1}{2}}. \tag{1.4}$$

In our second example we consider a situation very similar to the first, but in which the increase in surface current U is made up, not by a vertical upwelling from below, but by a horizontal inflow from the sides. The results are strikingly different. Although the variation in wave-number is the same as in (1.1), the variation in amplitude is now given by

$$\frac{1}{a}\frac{\partial a}{\partial x} = -\frac{c+U}{(c+2U)^2}\frac{\partial U}{\partial x}. \tag{1.5}$$

This is accounted for by including in the energy balance the advection of wave energy by the transverse current V, as well as the work done against the corresponding stress component (equation (6.4)). The amplitude a is now found to be

$$a \propto [(c+2U)/c]^{-\frac{1}{2}}, \tag{1.6}$$

which is a weaker variation than in the previous case.

The appropriate generalization of the equation of energy balance is shown to be

$$\nabla . [E(\mathbf{c}_g + \mathbf{U})] + \tfrac{1}{2}S_{ij}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) = 0, \tag{1.7}$$

where S_{ij} denotes the radiation stress tensor. In § 8 this is applied to a third example, that of waves crossing a shearing current obliquely. The changes in wavelength and direction of propagation θ are as found by Johnson (1947), but the law governing the wave amplitude is shown to be

$$a \propto (\sin 2\theta)^{-\frac{1}{2}}, \tag{1.8}$$

which differs from Johnson's result.

2. Two-dimensional current: an asymptotic solution

In this section we shall obtain a formal solution for surface waves on a non-uniform current $\mathbf{U}(x)$ which has no transverse component. The solution is to be valid when

$$\sigma^{-1}\frac{\partial U}{\partial x} \ll 1, \tag{2.1}$$

where σ is the wave frequency; in other words, the change in stream velocity U over one wavelength L (that is, $L\partial U/\partial x$) is assumed small compared with the wave velocity $L\sigma/2\pi$.

General equations

It will be supposed that the velocity field $\mathbf{u}$ is irrotational:

$$\mathbf{u} = \nabla\phi; \tag{2.2}$$

that the fluid is incompressible:

$$\nabla . \mathbf{u} = \nabla^2\phi = 0; \tag{2.3}$$

and that viscous effects are negligible. Then we have Bernoulli's integral

$$\frac{p}{\rho}+gz+\tfrac{1}{2}\mathbf{u}^2+\frac{\partial\phi}{\partial t}=C, \tag{2.4}$$

where p, ρ, g denote the pressure, density and acceleration of gravity, and z is the vertical co-ordinate, directed upwards. C is a constant. If $z=\zeta$ is the equation of the free surface, then for the two boundary conditions there are the kinematical condition

$$\frac{\partial\zeta}{\partial t}+\left(\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}-\frac{\partial\phi}{\partial z}\right)_{z=\zeta}=0, \tag{2.5}$$

and the condition of constant pressure, which by (2·4) may be written

$$g\zeta+\left(\tfrac{1}{2}\mathbf{u}^2+\frac{\partial\phi}{\partial t}\right)_{z=\zeta}=C. \tag{2.6}$$

It is convenient to replace these last two equations by conditions to be satisfied at the mean surface level $z=0$; this may be done by assuming the potential ϕ to be analytic and by expanding in a Taylor series in z:

$$\left.\begin{aligned}\frac{\partial\zeta}{\partial t}+\left(\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}-\frac{\partial\phi}{\partial z}\right)_{z=0}+\zeta\left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}-\frac{\partial\phi}{\partial z}\right)\right]_{z=0}+\ldots&=0,\\ g\zeta+\left(\tfrac{1}{2}\mathbf{u}^2+\frac{\partial\phi}{\partial t}\right)_{z=0}+\zeta\left[\frac{\partial}{\partial z}\left(\tfrac{1}{2}\mathbf{u}^2+\frac{\partial\phi}{\partial t}\right)\right]_{z=0}+\ldots&=C.\end{aligned}\right\} \tag{2.7}$$

Lastly, we assume that the waves are effectively in deep water, so that as $z\to-\infty$ the periodic part of the motion tends to zero.

Form of the solution

We seek a solution having the character of a time-periodic wave-motion superimposed upon a non-uniform steady flow. Let us then substitute

$$\left.\begin{aligned}\phi&=U_0x+(\alpha\phi_{10}+\beta\phi_{01})+(\alpha^2\phi_{20}+\alpha\beta\phi_{11}+\beta^2\phi_{02})+\ldots,\\ \zeta&=\qquad\;\;(\alpha\zeta_{10}+\beta\zeta_{01})+(\alpha^2\zeta_{20}+\alpha\beta\zeta_{11}+\beta^2\zeta_{02})+\ldots,\end{aligned}\right\} \tag{2.8}$$

where U_0 is a steady uniform velocity, the velocity of the stream at $x=0$; ϕ_{01} represents a steady non-uniform current, zero at $x=0$; and ϕ_{10} represents an undisturbed surface wave; α and β are arbitrary small parameters proportional to wave steepness and to the velocity gradient of the current respectively. The terms $\alpha^2\phi_{20}$, etc., are correction terms of higher order, necessary in order to satisfy the boundary conditions at the free surface. We are particularly interested in evaluating the second-order term $\alpha\beta\phi_{11}$, which is the lowest-order interaction potential between the waves and the current.

It may be worth remarking that to eliminate the uniform current U_0 by taking axes moving with velocity U_0 would not be convenient, since in the new frame of reference the motion would no longer be perfectly periodic in time. This is because the modified wavelength is generally a function of x, as will be seen below. Clearly the choice of axes must be made so as to correspond with the physical problem; if the source of the wave-motion is periodic this determines the appropriate frame of reference uniquely.

Retaining terms as far only as $\alpha\beta$, we have from (2.8)

$$\left.\begin{aligned}\nabla\phi &= (U_0, 0, 0)+\alpha\nabla\phi_{10}+\beta\nabla\phi_{01}+\alpha\beta\,\nabla\phi_{11},\\ \frac{\partial\phi}{\partial t} &= \alpha\frac{\partial\phi_{10}}{\partial t}+\alpha\beta\frac{\partial\phi_{11}}{\partial t},\\ \zeta &= \alpha\zeta_{10}+\beta\zeta_{01}+\alpha\beta\zeta_{11},\end{aligned}\right\} \tag{2.9}$$

and so

$$\left.\begin{aligned}\tfrac{1}{2}\mathbf{u}^2 &= \tfrac{1}{2}U_0^2+U_0\left(\alpha\frac{\partial\phi_{10}}{\partial x}+\beta\frac{\partial\phi_{01}}{\partial x}\right)\\ &\quad+\alpha\beta\left(U_0\frac{\partial\phi_{11}}{\partial x}+\frac{\partial\phi_{10}}{\partial x}\frac{\partial\phi_{01}}{\partial x}+\frac{\partial\phi_{10}}{\partial y}\frac{\partial\phi_{01}}{\partial y}+\frac{\partial\phi_{10}}{\partial z}\frac{\partial\phi_{01}}{\partial z}\right)+\ldots,\\ \frac{\partial}{\partial z}(\tfrac{1}{2}\mathbf{u}^2) &= U_0\left(\alpha\frac{\partial^2\phi_{10}}{\partial x\partial z}+\beta\frac{\partial\phi_{01}}{\partial x\partial z}\right)+\ldots.\end{aligned}\right\} \tag{2.10}$$

Substitution in (2.7) shows at once that

$$C = \tfrac{1}{2}U_0^2. \tag{2.11}$$

The terms in α now give

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\right)\zeta_{10}-\frac{\partial\phi_{10}}{\partial z} &= 0,\\ g\zeta_{10}+\left(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\right)\phi_{10} &= 0,\end{aligned}\right\} \tag{2.12}$$

to be satisfied at $z = 0$. On eliminating ζ_{10}, we have

$$\left(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\right)^2\phi_{10}+g\frac{\partial\phi_{10}}{\partial z} = 0 \quad (z = 0). \tag{2.13}$$

If we choose for ϕ_{10} the wave potential

$$\phi_{10} = A\,e^{k_0 q-i\sigma t}, \tag{2.14}$$

where A and k_0 are constants and

$$q = z+ix, \tag{2.15}$$

then ϕ_{10} satisfies Laplace's equation (2.3), and from (2.13)

$$(\sigma-U_0k_0)^2 = gk_0. \tag{2.16}$$

Introducing the reference velocity

$$c_0 = \surd(g/k_0) \tag{2.17}$$

and the non-dimensional parameter

$$\gamma = U_0/c_0 \tag{2.18}$$

we have from (2.16)

$$\sigma = c_0k_0(1+\gamma). \tag{2.19}$$

(To ensure continuity as γ (or U_0) tends to zero, we have adopted the positive sign in the square root.) From (2.12) and (2.19), we have also

$$\zeta_{10} = -\frac{1}{g}\left[\left(\frac{\partial}{\partial t}+\gamma c_0\frac{\partial}{\partial x}\right)\phi_{10}\right]_{z=0} = \frac{i}{c_0}(\phi_{10})_{z=0}. \tag{2.20}$$

Returning to equation (2.7), we see that the terms in β give equations for ϕ_{01} formally identical with (2.12) except that the time derivatives are now zero:

$$\left.\begin{aligned} U_0 = \frac{\partial \zeta_{01}}{\partial x} - \frac{\partial \phi_{01}}{\partial z} &= 0 \\ g\zeta_{01} + U_0 \frac{\partial \phi_{01}}{\partial x} &= 0 \end{aligned}\right\} \quad (z = 0), \tag{2.21}$$

whence

$$U_0^2 \frac{\partial^2 \phi_{01}}{\partial x^2} + g \frac{\partial \phi_{01}}{\partial z} = 0 \quad (z = 0). \tag{2.22}$$

We require a potential ϕ_{01} to represent a steady flow having no transverse component $\partial\phi_{01}/\partial y$, which satisfies Laplace's equation, and also the condition $(\partial\phi_{01}/\partial x)_{x=0} = 0$. Such a potential is

$$\phi_{01} = c_0 k_0 (x^2 - z^2) + D c_0 z, \tag{2.23}$$

where D is a constant to be determined. From (2.22),

$$D = -2\gamma^2. \tag{2.24}$$

Therefore

$$\left.\begin{aligned} \phi_{01} &= c_0 k_0 (x^2 - z^2) - 2\gamma^2 c_0 z, \\ \zeta_{01} &= -2\gamma x; \end{aligned}\right\} \tag{2.25}$$

also

$$\frac{\partial U}{\partial x} = \beta \frac{\partial^2 \phi_{01}}{\partial x^2} = 2\beta c_0 k_0 = 2\beta\sigma(1+\gamma)^{-1} \tag{2.26}$$

in accordance with (2.1), since β is assumed small.

The interaction potential

In equations (2.7) the terms in $\alpha\beta$ yield

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right)\zeta_{11} - \frac{\partial \phi_{11}}{\partial z} + \left(\frac{\partial \phi_{10}}{\partial x}\frac{\partial \zeta_{01}}{\partial x} + \frac{\partial \phi_{01}}{\partial x}\frac{\partial \zeta_{10}}{\partial x}\right) - \left(\zeta_{10}\frac{\partial^2 \phi_{01}}{\partial z^2} + \zeta_{01}\frac{\partial^2 \phi_{10}}{\partial z^2}\right) &= 0, \\ g\zeta_{11} + \left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right)\phi_{11} + \left(\frac{\partial \phi_{10}}{\partial x}\frac{\partial \phi_{01}}{\partial x} + \frac{\partial \phi_{10}}{\partial z}\frac{\partial \phi_{01}}{\partial z}\right) + \zeta_{01}\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right)\frac{\partial \phi_{10}}{\partial z} &= 0, \end{aligned}\right\} \tag{2.27}$$

to be satisfied when $z = 0$. (Note that $\partial^2\phi_{01}/\partial t\,\partial z = \partial^2\phi_{01}/\partial x\,\partial z = 0$.) From these equations ζ_{11} may be eliminated by applying the operator $g^{-1}(\partial/\partial t + U_0\partial/\partial x)$ to the second equation and then subtracting the first. Without substituting explicit expressions for ϕ_{10}, ϕ_{01}, ζ_{10} and ζ_{01} but using (2.12) and (2.21) and the fact that $\partial\phi_{10}/\partial z = k_0\phi_{10}$, we obtain*

$$\begin{aligned} &\frac{1}{g}\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right)^2 \phi_{11} + \frac{\partial \phi_{11}}{\partial z} \\ &\quad = 2\left(\frac{\partial \phi_{10}}{\partial x}\frac{\partial \zeta_{01}}{\partial x} + \frac{\partial \phi_{01}}{\partial x}\frac{\partial \zeta_{10}}{\partial x} + k_0\zeta_{10}\frac{\partial \phi_{01}}{\partial z}\right) - \zeta_{10}\frac{\partial^2 \phi_{01}}{\partial z^2} \quad (z = 0). \end{aligned} \tag{2.28}$$

Now, after substitution from (2.20) and (2.25), the right-hand side of this equation becomes

$$[2ik_0(1 - 2\gamma - 2\gamma^2) - 4k_0^2 x]\,\phi_{10}. \tag{2.29}$$

* In the calculation of ϕ_{11} the complex form of ϕ_{10} can be used, since on the right of (2.28) only products involving ϕ_{10} and ϕ_{01} occur, and ϕ_{01} is real.

As a trial solution let us write

$$\phi_{11} = i(k_1 q + l_1^2 q^2)\,\phi_{10}, \tag{2.30}$$

where $q = z + ix$ and k_1, l_1 are constants to be determined. Then the left-hand side of (2.28), when $z = 0$, reduces to

$$[(1+2\gamma)\,i(k_1 + 2il_1^2 x) - 2i\gamma^2 l_1^2/k_0]\,\phi_{10}. \tag{2.31}$$

On equating coefficients of ϕ_{10} and $x\phi_{10}$ in (2.29) and (2.31) we obtain

$$\left.\begin{aligned} k_1 &= 2k_0\frac{1-4\gamma^2-4\gamma^3}{(1+2\gamma)^2}, \\ l_1^2 &= 2k_0^2\frac{1}{(1+2\gamma)}. \end{aligned}\right\} \tag{2.32}$$

The second of equations (2.27) also gives

$$g\zeta_{11} = [(\gamma k_1 + 2\gamma^2 k_0) - ix(2k_0^2 + 2\gamma k_0^2 - 2\gamma l_1^2 + k_0 k_1) + k_0 l_1^2 x^2]\,c_0\phi_{10}. \tag{2.33}$$

This then is a formal solution of our problem.

Interpretation

Combining (2.33) with (2.20), we have

$$\alpha\zeta_{10} + \alpha\beta\zeta_{11} = \alpha\zeta_{10}\left[1 - i\beta\left(\frac{\gamma k_1}{k_0} + 2\gamma + l_1^2 x^2\right) - 2\beta k_0 x\left(1 + \gamma - \frac{\gamma l_1^2}{k_0^2} + \frac{k_1}{2k_0}\right)\right]. \tag{2.34}$$

Correct to order β, this expression may be written

$$\alpha\zeta_{10} + \alpha\beta\zeta_{11} = \alpha i\left(\frac{A}{c_0}\right)\exp\left\{i\left[k_0 x - \beta\left(\frac{\gamma k_1}{k_0} + 2\gamma + l_1^2 x^2\right)\right]\right\} \times\left[1 - 2\beta k_0 x\left(1 + \gamma - \frac{\gamma l_1^2}{k_0^2} + \frac{k_1}{2k_0}\right)\right]. \tag{2.35}$$

Now this represents a wave of slowly varying amplitude and wavelength. The local wave-number k is given by the x-derivative of the exponent:

$$k = k_0 - 2\beta l_1^2 x. \tag{2.36}$$

The proportional rate of change of the wave-number at $x = 0$ is therefore

$$\left(\frac{1}{k}\frac{\partial k}{\partial x}\right)_{x=0} = -\frac{2\beta l_1^2}{k_0} = -\frac{4\beta k_0}{1+2\gamma} \tag{2.37}$$

by (2.32). From (2.19) this may be written

$$\left(\frac{1}{k}\frac{\partial k}{\partial x}\right)_{x=0} = -\frac{2}{1+2\gamma}\frac{1}{c_0}\frac{\partial U}{\partial x}. \tag{2.38}$$

The amplitude a of the wave is given by

$$a = \alpha\left(\frac{A}{c_0}\right)\left[1 - 2\beta k_0 x\left(1 + \gamma - \frac{\gamma l_1^2}{k_0^2} + \frac{k_1}{2k_0}\right)\right], \tag{2.39}$$

so that the proportional rate of increase is

$$\left(\frac{1}{a}\frac{\partial a}{\partial x}\right)_{x=0} = -2\beta k_0\left(1+\gamma-\frac{\gamma l_1^2}{k_0^2}+\frac{k_1}{2k_0}\right) = -2\beta k_0\frac{2+3\gamma}{(1+2\gamma)^2} \tag{2.40}$$

by equation (2.32), or, from (2.26),

$$\left(\frac{1}{a}\frac{\partial a}{\partial x}\right)_{x=0} = -\frac{2+3\gamma}{(1+2\gamma)^2}\frac{1}{c_0}\frac{\partial U}{\partial x}. \tag{2.41}$$

The mean surface level

Equation (2.25) shows that there is a small change in the mean surface level given by

$$\beta\zeta_{01} = -2\beta\gamma x, \tag{2.42}$$

corresponding to a mean gradient $-2\beta\gamma$, as we should expect in a non-uniform flow. The additional terms $\alpha\zeta_{10}+\alpha\beta\zeta_{11}$ give no change in the mean level. Therefore to order $\alpha\beta$ the mean surface level is unaffected; only at higher approximations is any change apparent.

3. A physical discussion

We have seen that the interaction between the waves and the current can be interpreted as a distortion of the waves, resulting in a change of wavelength and amplitude. In this section we shall try to interpret these changes on the basis of rough physical reasoning.

As before, we denote by σ the angular frequency of the waves (constant over the whole field of motion) and by a, k, U, c the local wave amplitude, wave-number, stream velocity, and wave velocity relative to the stream. Our object is to obtain a, k and c as functions of U and of their values a_0, k_0, U_0, c_0 in some fixed plane $x = 0$.

The change in wavelength

Consider first the variation in wavelength. Now, the apparent velocity of the waves relative to a fixed plane x = constant is equal to $(c+U)$. The apparent angular frequency of the waves is therefore $k(c+U)$. But by hypothesis this quantity is equal to σ at all points, so that

$$k(c+U) = \sigma = k_0(c_0+U_0). \tag{3.1}$$

Thus

$$\frac{k}{k_0} = \frac{c_0+U_0}{c+U}. \tag{3.2}$$

But the waves being in deep water we expect that their velocities c, c_0 relative to the current will be given by the classical formulae

$$c^2 = g/k, \quad c_0^2 = g/k_0. \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\frac{c^2}{c_0^2} = \frac{k_0}{k} = \frac{c+U}{c_0+U_0} = \frac{1}{1+\gamma}\left(\frac{c}{c_0}+\frac{U}{c_0}\right), \tag{3.4}$$

where $\gamma = U_0/c_0$ as before. On differentiation with respect to x, we have

$$\frac{2c}{c_0^2}\frac{\partial c}{\partial x} = \frac{1}{1+\gamma}\frac{1}{c_0}\left(\frac{\partial c}{\partial x}+\frac{\partial U}{\partial x}\right), \tag{3.5}$$

and hence at $x = 0$, where $c = c_0$,

$$\frac{\partial c}{\partial x} = \frac{1}{1+2\gamma}\frac{\partial U}{\partial x}. \tag{3.6}$$

Since by (3.4) k varies as c^{-2}, we have (by logarithmic differentiation)

$$\frac{1}{k}\frac{\partial k}{\partial x} = -\frac{2}{c}\frac{\partial c}{\partial x} = -\frac{2}{1+2\gamma}\frac{1}{c}\frac{\partial U}{\partial x} \tag{3.7}$$

in agreement with (2.38).

It will be seen that equation (3.4) is a quadratic in c/c_0, and has the solution

$$\frac{c}{c_0} = \frac{1}{2(1+\gamma)}\left\{1+\sqrt{\left[1+\frac{4(1+\gamma)\,U}{c_0}\right]}\right\} \tag{3·8}$$

(see Unna (1942), for the case $\gamma = 0$). In the square root, the positive sign has been taken to ensure continuity as $x \to 0$. It is interesting to note that no solution can exist when

$$1+\frac{4(1+\gamma)\,U}{c_0} < 0, \tag{3.9}$$

or

$$-U > \frac{c_0}{4(1+\gamma)}, \tag{3.10}$$

that is to say, when the stream velocity is in the opposite direction and exceeds in magnitude about one-quarter of the initial phase velocity of the waves. At the critical point, when the radical vanishes, equation (3·8) shows that

$$\frac{c}{c_0} = \frac{1}{2(1+\gamma)}, \tag{3.11}$$

and so

$$\frac{U}{c} = \frac{U}{c_0}\frac{c_0}{c} = -\tfrac{1}{2}. \tag{3.12}$$

In other words the stream velocity becomes equal and opposite to the local group-velocity $\frac{1}{2}c$; the wave energy can no longer be propagated against the stream. We shall see below that the waves tend to break before this point is reached. From (3.8) we have also

$$\frac{k}{k_0} = \left(\frac{c}{c_0}\right)^{-2} = \left\{\frac{2(1+\gamma)}{1+\sqrt{[1+4(1+\gamma)\,U/c_0]}}\right\}^2. \tag{3.13}$$

The changes in wave amplitude

The change in wave amplitude is interesting, for it enables us to decide between various conflicting hypotheses.

It was shown in I that if waves of amplitude a are propagated over a stream of *uniform* velocity U, the mean rate of energy transfer across a plane x = const. is given (to order a^2) by

$$\bar{R}_x = E(c_g+U)+S_x U+\tfrac{3}{2}EU^2/c+\tfrac{1}{2}\rho h U^3, \tag{3.14}$$

where E denotes the wave energy per unit horizontal area:

$$E = \tfrac{1}{2}\rho g a^2; \tag{3.15}$$

c_g denotes the group-velocity of the waves; in 'deep' water,

$$c_g = \tfrac{1}{2}c = \tfrac{1}{2}\sigma/k; \tag{3.16}$$

h is the mean depth of the stream, and S_x is defined by

$$S_x = E\left(\frac{2c_g}{c} - \frac{1}{2}\right). \tag{3.17}$$

The first term $E(c_g + U)$ on the right-hand side of (3.14) represents simply the transfer of wave energy by the group-velocity plus the stream velocity, and is to be expected. The last two terms may be written together as $\frac{1}{2}\rho h U'^3$, where

$$U' = U + E/\rho c h \tag{3.18}$$

represents the mean stream velocity modified by the presence of the mass transport. The intermediate term $S_x U$ has been discussed in I. It represents a kind of coupling between the waves and the current. By analogy with the Reynolds stress, S_x has been called the 'radiation stress'.

Now, in the present problem of waves on a non-uniform stream, let us suppose that the transfer of total energy is given with sufficient accuracy by equation (3.14) and further that between the planes $x = 0$ and $x = \text{const.}$ there is no reflexion of wave energy. It follows then that

$$\bar{R}_x = \bar{R}_0 = \text{const.} \tag{3.19}$$

and so

$$\frac{\partial}{\partial x}\bar{R}_x = 0. \tag{3.20}$$

Equation (3.20) is merely an expression of the conservation of the energy, when dissipative mechanisms are ignored. However, it is possible to regard it as the sum of two equations, one representing the balance of wave energy and the other the balance of mean flow energy.

For the exact form of this division, no unique answer is given by physical intuition. (At least *our* initial intuition, as well as that of Unna (1942), Evans (1955), Suthons (1945), Groen & Dorrestein (1958) and Drent (1959) yielded results which sometimes differed from one another but which were all, it as appears, incorrect.) Now, however, we have an arbiter for conflicts of intuition, for the correct division of (3.20) must yield results consistent with § 2.

The first five of the authors just named made the assumption that there was no interchange of energy between waves and current and thus obtained

$$\frac{\partial}{\partial x}[E(c_g + U)] = 0. \tag{3.21}$$

It is clear both from the results of I and from § 2 of the present paper that this assumption cannot be correct.

One might then argue that all the terms dependent on E belong properly to the wave-energy equation, and write

$$\frac{\partial}{\partial x}\left[E(c_g+U)+S_x U+\frac{\frac{3}{2}EU^2}{c}\right] = 0, \tag{3.22}$$

or, since the last term may be included with the mean flow,

$$\frac{\partial}{\partial x}[E(c_g+U)+S_x U] = 0. \tag{3.23}$$

Each of these equations (3.21), (3.22) and (3.23) yields results in conflict with § 2.

If, on the other hand, it is argued that the effect of the current variation on the wave energy is through the work done by the rate of strain against the radiation stress, then we have

$$\frac{\partial}{\partial x}[E(c_g+U)]+S_x\frac{\partial U}{\partial x} = 0. \tag{3.24}$$

Thus, in deep water,

$$\frac{\partial}{\partial x}[E(\tfrac{1}{2}c+U)]+\tfrac{1}{2}E\frac{\partial U}{\partial x} = 0. \tag{3.25}$$

Carrying out the differentiation at $x = 0$, where $U = \gamma c$, and using equation (3.6), we have

$$\frac{\partial E}{\partial x}[\tfrac{1}{2}c(1+2\gamma)]+E\left[\frac{1}{2(1+2\gamma)}+\frac{3}{2}\right]\frac{\partial U}{\partial x} = 0, \tag{3.26}$$

whence

$$\left(\frac{1}{E}\frac{\partial E}{\partial x}\right)_{x=0} = -\frac{4+6\gamma}{(1+2\gamma)^2}\frac{1}{c}\frac{\partial U}{\partial x}, \tag{3.27}$$

or, since E is proportional to a^2,

$$\left(\frac{1}{a}\frac{\partial a}{\partial x}\right)_{x=0} = -\frac{2+3\gamma}{(1+2\gamma)^2}\frac{1}{c}\frac{\partial U}{\partial x} \tag{3.28}$$

in exact agreement with equation (2.41).

It appears then that the correct assumption to make is equation (3.24), rather than the alternatives (3.21) to (3.23). We interpret this as follows:

In a non-uniform current the energy of the waves may be regarded as being transported with the group-velocity plus stream velocity, provided in addition we suppose that the mean stream does work on the waves at a rate $S_x\partial U/\partial x$ per unit distance, where S_x is the radiation stress. Equation (3.24) is then the expression of the energy balance for the waves.

An integral for the wave amplitude

An exact integral of equation (3.25) is

$$E(\tfrac{1}{2}c+U)\,c = \text{const.}, \tag{3.29}$$

for on differentiating the above and dividing by c, we have

$$\frac{\partial}{\partial x}[E(\tfrac{1}{2}c+U)]+E(\tfrac{1}{2}c+U)\frac{1}{c}\frac{\partial c}{\partial x} = 0, \tag{3.30}$$

which by (3.6) is equivalent to (3.25). From equation (3.29) we deduce

$$\frac{E}{E_0} = \frac{c_0(c_0+2U_0)}{c(c+2U)}, \tag{3.31}$$

and thence

$$\frac{a}{a_0} = \left[\frac{c_0(c_0+2U_0)}{c(c+2U)}\right]^{\frac{1}{2}}. \tag{3.32}$$

This law of amplification is illustrated by curve (1) of figure 1. At the critical point, where $U = -\frac{1}{2}c$, the amplification of the waves becomes theoretically infinite. In practice the waves may be expected to break, but the present small-amplitude theory becomes inapplicable before this point is reached.

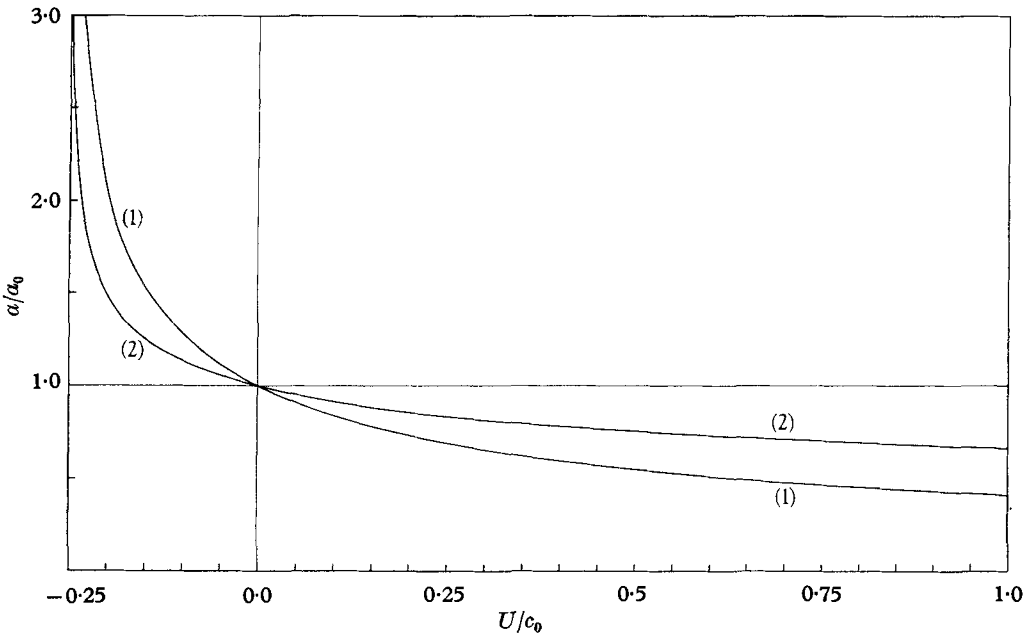

FIGURE 1. The amplification factor a/a_0 for waves on a current U in the direction of wave propagation: (1) with vertical upwelling from below; (2) with horizontal inflow from the sides. [a_0 and c_0 denote the values of a and c when $U = 0$.]

We are indebted to a referee for pointing out that a result similar to (3.29) was derived in an unpublished report by Kreisel (c. 1944, pp. 23–24). Kreisel started from the energy integral

$$\overline{\int_{-\infty}^{\zeta} \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x}\,dz} = \text{const.},$$

and assumed that (in our notation)

$$\phi = Ux - ac\,e^{kz}\cos(kx-\sigma t),$$
$$\zeta = a\sin(kx-\sigma t),$$

where

$$c^2 = gk = (\sigma - kU)^2.$$

Substituting in the integral and treating U, c, a and k as constants during differentiation, one finds eventually

$$\tfrac{1}{2}\sigma a^2 c(c+2U) = \text{const.}$$

(higher powers of a being neglected). Since σ is constant this agrees with (3.29), and indeed provides a physical explanation of that equation. The crux of Kreisel's argument is the assumption that $\partial\phi/\partial t$ contains no constant terms proportional to a^2. This is true for deep water, but not in water of finite depth.

The rules for the variation of wave-number and wave amplitude expressed by (3.13) and (3.32) may be regarded as generalizations of the results found in § 2, the only additional assumptions being that $k^{-1}\partial k/\partial x$ and $a^{-1}\partial a/\partial x$ depend on the local values of U, c and $\partial U/\partial x$ and are linearly proportional to $\partial U/\partial x$.

The analysis of § 2 is correct as far as the first power of $\beta k_0 x$ only. In order to verify that (3.13) and (3.32) are correct to this order we write

$$\frac{U-U_0}{U_0} = \epsilon, \tag{3.33}$$

so that

$$\frac{U}{U_0} = 1+\epsilon, \quad \frac{U}{c_0} = \gamma(1+\epsilon). \tag{3.34}$$

Substituting in equations (3.13) and (3.32) and neglecting ϵ^2 we find, after some reduction,

$$\left.\begin{aligned} \frac{k}{k_0} &= 1-\frac{2\gamma}{1+2\gamma}\epsilon, \\ \frac{a}{a_0} &= 1-\frac{\gamma(2+3\gamma)}{(1+2\gamma)^2}\epsilon, \end{aligned}\right\} \tag{3.35}$$

of which (2.36) and (2.39) will be seen to be special cases.

4. An application to tidal currents

As an example of the application of the general formulae, and as an independent check, we apply the formulae to the case of surface waves on a tidal current, for which a solution was obtained independently in I.

A short wave of mean amplitude a_1, mean wave-number k_1 and frequency σ is assumed to be superposed upon a long (shallow-water) wave of amplitude a_2, wave-number k_2 and frequency σ_2, travelling in the same direction as the first. The conditions of the problem are that

$$\left.\begin{aligned} \sigma_2/\sigma_1 &= \lambda \ll 1, \\ k_2 h &= \mu \ll 1, \end{aligned}\right\} \tag{4.1}$$

where h is the mean depth of water; also

$$e^{-k_1 h} \ll 1. \tag{4.2}$$

This last assumption ensures that the short waves are effectively in deep water, so that

$$\sigma_1 = (gk_1)^{\frac{1}{2}}, \quad \sigma_2 = (gh)^{\frac{1}{2}}k_2; \quad k_1 h = (\mu/\lambda)^2. \tag{4.3}$$

In the case of tidal currents both λ and μ may be of order 10^{-4} in a typical case, but the ratio μ/λ, $= (k_1 h)^{\frac{1}{2}}$, need not be greater than about 2 in order for the condition (4.2) to be satisfied.

Now let us reduce the long wave to a steady current $\mathbf{U}$ by superposing on the whole system a uniform stream $-(gh)^{\frac{1}{2}}$. Choosing the origin of x at a node of the longer wave, we have

$$U = -(gh)^{\frac{1}{2}}\left(1-\frac{a_2}{h}\sin k_2 x\right)+O(a^2). \tag{4.4}$$

At $x = 0$ the stream velocity and the velocity of the short waves are given by

$$U_0 = -(gh)^{\frac{1}{2}}; \quad c_0 = (g/k_1)^{\frac{1}{2}}. \tag{4.5}$$

Thus,
$$\gamma = \frac{U_0}{c_0} = -(k_1 h)^{\frac{1}{2}} = -\frac{\mu}{\lambda}. \tag{4.6}$$

Also
$$\epsilon = \frac{U - U_0}{U_0} = -\frac{a_2}{h}\sin k_2 x. \tag{4.7}$$

On substitution in (3.35), we find

$$\left.\begin{aligned} \frac{k}{k_0} &= 1 - \frac{2\mu}{\lambda - 2\mu}\frac{a_2}{h}\sin k_2 x, \\ \frac{a}{a_0} &= 1 - \frac{(2\lambda - 3\mu)\,\mu}{(\lambda - 2\mu)^2}\frac{a_2}{h}\sin k_2 x, \end{aligned}\right\} \tag{4.8}$$

in agreement with equations (2.56) and (2.57) of I.* When μ/λ is sufficiently large, then

$$\left.\begin{aligned} \frac{k}{k_0} &= 1 + \frac{a_2}{h}\sin k_2 x, \\ \frac{a}{a_0} &= 1 + \frac{3}{4}\frac{a_2}{h}\sin k_2 x. \end{aligned}\right\} \tag{4.9}$$

5. Waves on a converging current: no upwelling

In the last three sections we have been concerned with an entirely two-dimensional motion in which the transverse component of the mean current was zero; the increase in the stream velocity with horizontal distance was made up by a compensating current upwelling from below. We now study a somewhat different situation in which the *vertical* component of current vanishes and the increase in the horizontal x-component U is compensated entirely by a horizontal in-flow V from the sides:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \tag{5.1}$$

The analysis for the asymptotic solution is identical with that in the previous case, § 2, as far as equation (2.22). Now, however, instead of the potential (2.23) we must choose a potential ϕ_{01} to represent a flow having zero vertical component, and satisfying the equation of continuity (5.1). We take

$$\phi_{01} = c_0 k_0 (x^2 - y^2) + D c_0 z, \tag{5.2}$$

and from (2.22) we see that the constant D has to be $-2\gamma^2$ as before. Thus,

$$\left.\begin{aligned} \phi_{01} &= c_0 k_0 (x^2 - y^2) - 2\gamma^2 c_0 z, \\ \zeta_{01} &= -2\gamma x, \end{aligned}\right\} \tag{5.3}$$

and (2.26) still applies.

In the equations (2.27) for the interaction potential, the additional terms all vanish identically, so that (2.28) is still valid; the only difference is that the last term $\zeta_{10}\partial^2\phi_{01}/\partial z^2$ vanishes, and so in place of (2.29) we have

$$[2ik_0(-2\gamma - 2\gamma^2) + 4k_0^2 x]\,\phi_{10}. \tag{5.4}$$

* In equation (2.57) of I, the second term in the curly bracket can be neglected, since $\lambda < \mu \ll 1$.

Now, on equating coefficients between (2.31) and (5.4), we find

$$\left.\begin{aligned} k_1 &= -4k_0\frac{\gamma+2\gamma^2+2\gamma^3}{(1+2\gamma)^2},\\ l_1^2 &= 2k_0^2\frac{1}{1+2\gamma}. \end{aligned}\right\} \tag{5.5}$$

Since the value of l_1^2 is still the same, equations (2.34) to (2.38) are still applicable and in particular (2.38) shows that we have the same rate of change of the wave-number k as in the previous case.

But, since k_1 has a different value, equation (2.41) must now be replaced by

$$\left(\frac{1}{a}\frac{\partial a}{\partial x}\right)_{x=0} = -\frac{1+\gamma}{(1+2\gamma)^2}\frac{1}{c_0}\frac{\partial U}{\partial x}, \tag{5.6}$$

showing that the change in *amplitude* of the waves is different from the previous case.

6. Physical interpretation

The current U along the x-axis being as in § 3, the changes in wave velocity and wave-number which were derived in that section (by arguments depending only on kinematical considerations) are still given by (3.6) and (3.7). This confirms what was found in § 5 concerning the change in wave-number.

The change in wave amplitude, however, must be related to the equation of energy transfer. Now it was found in I that in the presence of a horizontal stream $\mathbf{U} = (U, V, 0)$ not necessarily in the x-direction, the mean transfer of energy across a vertical plane whose normal is $\mathbf{n} = (l, m, 0)$ is given by

$$\bar{R} = E(\mathbf{c}_g+\mathbf{U}).\mathbf{n}+\mathbf{U}.\mathbf{S}.\mathbf{n}+\tfrac{1}{2}\rho h\mathbf{U}'^2(\mathbf{U}'.\mathbf{n}), \tag{6.1}$$

where $\mathbf{c}_g$ denotes the vector group-velocity, $\mathbf{U}'$ denotes the stream velocity as modified by the mass-transport and $\mathbf{S}$ is a stress tensor. If the x-direction is the direction of wave propagation, then $\mathbf{c}_g = (c_g, 0, 0)$, $\mathbf{U}' = \mathbf{U}+(E/\rho ch, 0, 0)$, and

$$\mathbf{S} = \begin{pmatrix} S_x & 0 & 0\\ 0 & S_y & 0\\ 0 & 0 & 0 \end{pmatrix}, \tag{6.2}$$

where S_x is given by (3.17) and

$$S_y = E\left(\frac{c_g}{c}-\frac{1}{2}\right). \tag{6.3}$$

Therefore a natural generalization of equation (3.14) is to assume

$$\nabla.[E(\mathbf{c}_g+\mathbf{U})]+\left[S_x\frac{\partial U}{\partial x}+S_y\frac{\partial V}{\partial y}\right] = 0. \tag{6.4}$$

In other words, the divergence of the energy flux is exactly compensated by work done by the mean current against the radiation stress. In deep water this becomes

$$\frac{\partial}{\partial x}[E(\tfrac{1}{2}c+U)]+\frac{\partial}{\partial y}[EV]+\tfrac{1}{2}E\frac{\partial U}{\partial x} = 0. \tag{6.5}$$

By the symmetry of the flow about the plane $y = 0$, we see that, on the x-axis, $\partial E/\partial y$ vanishes identically, and so making use of (5.1) we have

$$\frac{\partial E}{\partial x}(\tfrac{1}{2}c+U)+\tfrac{1}{2}E\left(\frac{\partial c}{\partial x}+\frac{\partial U}{\partial x}\right) = 0. \tag{6.6}$$

On substituting for $\partial c/\partial x$ from (3.6), we find

$$\left(\frac{1}{E}\frac{\partial E}{\partial x}\right)_{x=0} = -\frac{2(1+\gamma)}{(1+2\gamma)^2}\frac{1}{c}\frac{\partial U}{\partial x}, \tag{6.7}$$

from which follows

$$\left(\frac{1}{a}\frac{\partial x}{\partial x}\right)_{x=0} = -\frac{1+\gamma}{(1+2\gamma)^2}\frac{1}{c}\frac{\partial U}{\partial x}, \tag{6.8}$$

in exact agreement with (5.7).

Equation (6.5) may also be written as

$$\frac{\partial}{\partial x}[E(\tfrac{1}{2}c+U)]-\tfrac{1}{2}E\frac{\partial U}{\partial x} = 0, \tag{6.9}$$

which has the integral

$$E(\tfrac{1}{2}c+U)/c = \text{const.}, \tag{6.10}$$

as may be verified in the same way as (3.28). Hence, in the present situation,

$$\frac{E}{E_0} = \frac{c(c_0+2U_0)}{c_0(c+2U)}, \tag{6.11}$$

and

$$\frac{a}{a_0} = \left[\frac{c(c_0+2U_0)}{c_0(c+2U)}\right]^{\frac{1}{2}}. \tag{6.12}$$

It will be seen that as the critical point is approached, $a/a_0 \to \infty$ as before.

The amplitude variation corresponding to equation (6.12) is shown in figure 1, curve (2), compared with the corresponding variation in the case of no lateral flow.

7. Waves on currents of arbitrary form

To generalize our previous results, we note that **S** is a Cartesian tensor of rank 2, which we may write S_{ij}; equation (6.2) gives S_{ij} in diagonal form, when referred to axes perpendicular and parallel to the local wave front.

The velocity gradients $\partial U/\partial x$ and $\partial V/\partial y$ are also components of the symmetric rate-of-strain tensor

$$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right), \tag{7.1}$$

and the generalization of the interaction term in the wave-energy equation is $S_{ij}\gamma_{ij}$, which is, of course, an invariant.

Hence the correct generalization of equation (6.4) for steady currents of arbitrary form is

$$\nabla.[E(\mathbf{c}_g+\mathbf{U})]+\tfrac{1}{2}S_{ij}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right) = 0. \tag{7.2}$$

For time-varying currents we assume

$$\frac{\partial E}{\partial t}+\nabla.[E(\mathbf{c}_g+\mathbf{U})]+\tfrac{1}{2}S_{ij}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right) = 0. \tag{7.3}$$

In the case of purely two-dimensional motion ($\partial/\partial y = 0$), this reduces to

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x}[E(\mathbf{c}_g+\mathbf{U})]+S_x\frac{\partial U}{\partial x} = 0, \tag{7.4}$$

an equation that was verified approximately in § 5 of I. In that paper it was not possible to distinguish between equation (7.4) and the same equation with $\partial(S_x U)/\partial x$ replacing $S_x \partial U/\partial x$, since the difference, $(\partial S_x/\partial x)\, U$, was negligibly small. However, the technique adopted in § 4 of the present paper, whereby the long wave was reduced to rest by superposing a finite negative velocity, removes the ambiguity in the final term.

Given the appropriate boundary conditions, equation (7.3) is generally sufficient to determine the variation in the wave-energy density E. From this the variation in wave amplitude may be deduced on the assumption that the relation between amplitude and energy-density is

$$E = \tfrac{1}{2}\rho g a^2(1+\dot{W}/2g), \tag{7.5}$$

where $\dot{W}$ denotes the vertical acceleration of a particle carried by the mean current.* (See § 4 of I.) For steady currents we have

$$\dot{W} = \kappa(U^2+V^2), \tag{7.6}$$

where κ is the curvature of the path of the particle. If $\dot{W}$ is small compared with g then we may take

$$E = \tfrac{1}{2}\rho g a^2, \tag{7.7}$$

as has been assumed throughout this paper.

It may be mentioned that some experiments have recently been performed by Hughes (1960) on the interaction of waves and shear flows. These he has analysed using an assumption equivalent to (7.2), and his results tend to confirm the theory.

8. Waves on a shearing current

As a final example we shall apply the general equation (7.2) to the interesting case of waves traversing a simple horizontal current with vertical axis of shear. This was previously considered by Johnson (1947) without taking into account the transfer of energy between the waves and the current.†

The stream velocity $(0, V, 0)$ is supposed to be everywhere parallel to the y-axis, and also

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial z} = 0. \tag{8.1}$$

The wavelength and amplitude of the waves are supposed also to be independent of y. The angle which the waves make locally with the x-axis is denoted by θ (see figure 2).

Purely kinematical considerations yield the following: since the wave-number in the y-direction ($k\sin\theta$) must be independent of x, we have

$$k\sin\theta = m, \tag{8.2}$$

* It is assumed that the current is nearly horizontal.

† Some of the results of this section were obtained by Drent (1959) who, adopting a different approach, was led to make an assumption equivalent to (7.2) in this case.

a constant. Since the apparent velocity of the waves at right-angles normal to their crests is $(c + V\sin\theta)$ and their wave-number is k, the apparent angular frequency of the waves relative to a fixed point is

$$k(c+V\sin\theta) = \sigma, \tag{8.3}$$

also a constant. Thirdly, we have the relation connecting local wave-number and velocity:

$$kc^2 = g. \tag{8.4}$$

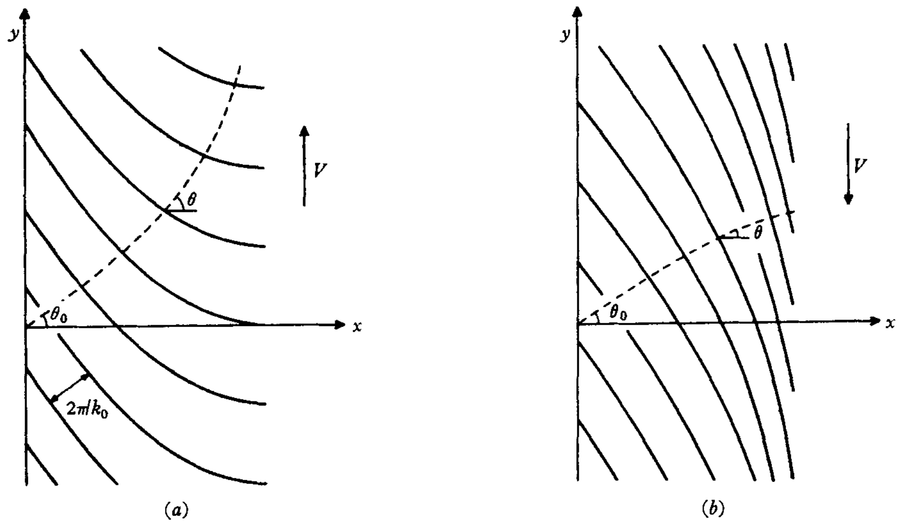

FIGURE 2. Definition diagram for waves on a shearing current, showing the qualitative effect of the current (*a*) when $V > 0$, (*b*) when $V < 0$.

From equation (8.3), by use of (8.4) and (8.2), it follows that

$$\frac{g}{c} + mV = \sigma, \tag{8.5}$$

or

$$c = \frac{g}{\sigma - mV}. \tag{8.6}$$

Then, from (7.4),

$$k = \frac{(\sigma - mV)^2}{g}, \tag{8.7}$$

and, from (7.2),

$$\sin\theta = \frac{mg}{(\sigma - mV)^2}. \tag{8.8}$$

If c_0, k_0, θ_0 denote the values of c, k, θ when the transverse velocity V vanishes, then we have

$$\left.\begin{aligned} \frac{c}{c_0} &= \frac{1}{1-(V/c_0)\sin\theta_0}, \\ \frac{k}{k_0} &= [1-(V/c_0)\sin\theta_0]^2, \\ \sin\theta &= \frac{\sin\theta_0}{[1-(V/c_0)\sin\theta_0]^2}. \end{aligned}\right\} \tag{8.9}$$

Since $\sin\theta$ cannot exceed unity, there is clearly an upper limit to V for which a solution exists:

$$V/c_0 \leqslant \frac{1-(\sin\theta_0)^{\frac{1}{2}}}{\sin\theta_0}. \tag{8.10}$$

At this upper limit θ becomes equal to $\frac{1}{2}\pi$, and the waves are totally reflected by the current.

On the other hand, for negative currents $V < 0$, there is no kinematic limit to V. However, as $V \to -\infty$, k becomes very large, that is to say the wavelength becomes very small (figure 2(*b*)). The angle θ approaches zero, that is, the direction of propagation becomes nearly normal to the current.

Now the vector group-velocity is given by

$$\mathbf{c}_g = \tfrac{1}{2}\mathbf{c} = (\tfrac{1}{2}c\cos\theta,\ \tfrac{1}{2}c\sin\theta). \tag{8.11}$$

Hence equation (7.2) becomes in this case

$$\frac{\partial}{\partial x}[E\,.\,\tfrac{1}{2}c\cos\theta]+\frac{\partial}{\partial y}[E(\tfrac{1}{2}c\sin\theta+V)]+\tfrac{1}{2}E\frac{\partial V}{\partial x}\cos\theta\sin\theta = 0. \tag{8.12}$$

Since all derivatives with respect to y vanish identically, we find, on substitution from (8.6) and (8.8),

$$\frac{\partial}{\partial x}\left(\frac{E\cos\theta}{\sigma-mV}\right)+\frac{Em\cos\theta}{(\sigma-mV)^2}\frac{\partial V}{\partial x} = 0, \tag{8.13}$$

of which the integral is

$$\frac{E\cos\theta}{(\sigma-mV)^2} = \text{const.}, \tag{8.14}$$

or, from (8.8),

$$E\cos\theta\sin\theta = \text{const.} \tag{8.15}$$

The relative amplification of the waves is therefore given by

$$\frac{a}{a_0} = \left(\frac{E}{E_0}\right)^{\frac{1}{2}} = \left(\frac{\sin 2\theta_0}{\sin 2\theta}\right)^{\frac{1}{2}}. \tag{8.16}$$

This ratio is shown graphically in figure 3 as a function of V/c_0, for various values of the initial angle θ_0.

Evidently the amplification of the waves becomes infinite both when $\theta \to 90^\circ$ and when $\theta \to 0$. In the first case the infinity is not significant: it is due to the fact that the ray-paths intersect, and the corresponding line $x = \text{const.}$ is a caustic. To the left of this line there are essentially two systems of waves, the incident and transmitted systems, while to the right of it there is a 'shadow zone'. In the neighbourhood of such a line the ordinary approximations of ray optics do not apply; a higher-order theory, generally involving Airy functions, must be used. One may expect that the wave amplitude in fact remains finite even in the neighbourhood of the critical line.

The second case, when $\theta \to 0$, corresponds to the limit $V \to -\infty$. In that case the infinity is genuine and is due mainly to the fact that the wavelength and wave velocity are so much reduced that, in order to maintain the energy flow in the x-direction, the amplitude must increase. In practice the waves may break; but for no finite velocity $V < 0$ is the ratio a/a_0 theoretically infinite.

We may note that it is possible for the component of stream velocity opposite to the waves to exceed the group-velocity:

$$\tfrac{1}{2}c + V\sin\theta < 0. \tag{8.17}$$

The waves are not thereby stopped, for the wave amplitude tends to be diminished by a lateral stretching of the wave crests.

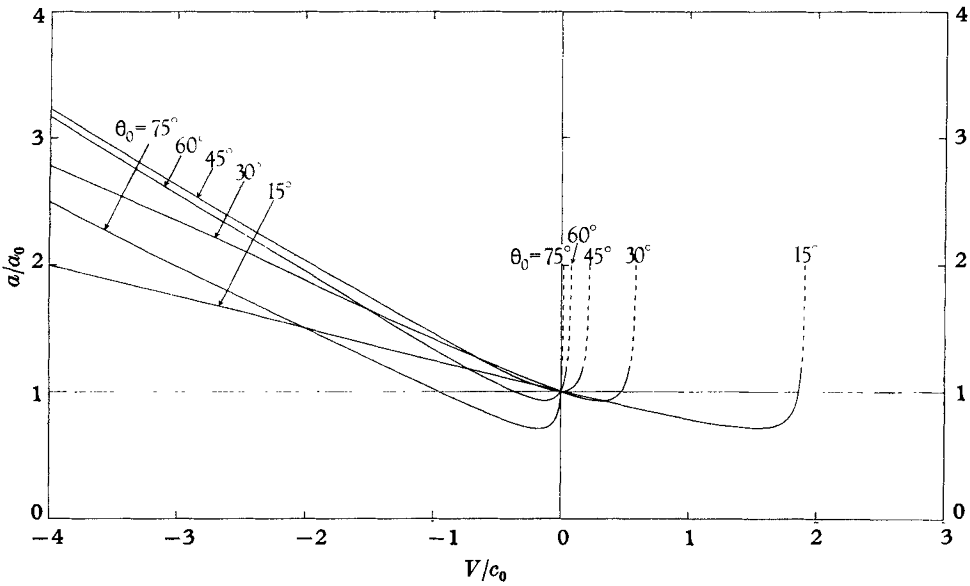

FIGURE 3. The amplification factor for waves crossing a shearing current V at an oblique angle θ, for various angles of entry θ_0.

9. Conclusions

The amplitude of surface waves on non-uniform currents is affected by a non-linear interaction between the waves and the components of the currents; the coupling terms are proportional to the radiation stresses, and the general equation governing the transfer of wave energy is equation (7.3).

Waves travelling on a non-uniform current U that varies in the direction of wave propagation undergo an amplification that is greater than previously supposed, and is dependent on whether the variation in current is made up by a small vertical upwelling from below or by a small horizontal inflow from the sides; this difference is illustrated by the two curves in figure 1.

The amplification of waves on a transverse shearing current has also been calculated. Here the interaction between waves and current also produces an amplification different from that obtained by neglecting the interaction terms.

The results show that the efficiency of a hydraulic or pneumatic breakwater should be affected not only by the surface currents directly opposing the waves but also by the transverse or vertical components of the secondary circulating flow, for these produce different effects on the wave steepening. The absolute limits to the wavelengths that can be transmitted are still set by Taylor's kinematical theory (1955). But for waves longer than the critical wavelength, whether breaking occurs must depend on the amplification factor. We suggest that differences in the secondary circulation may account for some of the anomalies in past experimental work, both on models and on prototypes.

Since the currents have been seen to do work on the waves, then we would expect the waves also to react on the currents. From (6.1), by conservation of the total energy, one would expect for steady currents

$$\nabla . [E(\mathbf{c}_g + \mathbf{U}) + \mathbf{S} . \mathbf{U} + (\tfrac{1}{2}\rho h \mathbf{U}'^2)\, \mathbf{U}'] = 0. \tag{9.1}$$

Hence, on subtracting (7.2) and using the fact that S_{ij} is symmetric, we have

$$\nabla . [(\tfrac{1}{2}\rho h \mathbf{U}'^2)\, \mathbf{U}'] + U_i \frac{\partial S_{ij}}{\partial x_j} = 0. \tag{9.2}$$

A fuller account of equation (9.2) will be given subsequently.

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